From last time:

Ex Suppose G has order 6. What isomorphism classes of groups G are possible?

Pf. Consider 3-Sylow subgroup $H_3 < G$.
$\Rightarrow H_3 = \{e, a, a^2\}$ (e = identity)

$N_3 = \#$ of 3-Sylow subgroups $\equiv 1 \mod 3$, must divide 2

$N_3 = 1$ or 2

$N_3 = 1$.

$\Rightarrow H_3$ is normal!

$$\Rightarrow \left|G/H_3\right| = 2 \Rightarrow \left(G/H_3\right) \cong \mathbb{Z}_2,$$

i.e. cyclic.

$\exists\, b \in G$ s.t. $G/H_3 = \{H_3, bH_3\}$, $bH_3 = H_3 b$

$\Rightarrow G\{e, a, a^2, b, ba, ba^2\}$

$$ab = \begin{cases} e \Rightarrow b = a^{-1} = a^2 \Rightarrow \times \\ a \Rightarrow b = e \ \times \\ a^2 \Rightarrow b = a \ \times \\ b \Rightarrow a = e \ \times \\ ba \\ ba^2 \end{cases}$$

$\boxed{\text{If } ab = ba}$, then

$b^k a^j = a^j b^k \ \forall j, k \in \mathbb{Z}$,

so G is abelian

$$\left(b^{k_1} a^{j_1} b^{k_2} a^{j_2} = b^{k_1+k_2} a^{j_1+j_2} = b^{k_2} a^{j_2} b^{k_1} a^{j_1}\right).$$

$\forall k_1, k_2, j_1, j_2$

$\therefore$ By FTFGAG,

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6.$$

$$b^k a^j = b^{k-1}(ba)a^{j-1} = b^{k-1} a b a^{j-1}$$
$$\cdots = b^{k-1} a^j b = \cdots a^j b^k$$

Next, if $ab = ba^2$, Then what is b^2

b^2 must be in H_3, so $b^2 = \begin{cases} e \text{ or} \\ a \text{ or} \\ a^2 \end{cases}$

$(bH_3)(bH_3) = H_3$
$\Rightarrow b^2 H_3 = H_3$
$\Rightarrow b^2 \in H_3$.

If $b^2 = a$, then

$ab = ba^2$

mult by a on right $\Rightarrow aba = b \overset{a=b^2}{\Rightarrow} b^5 = b$

$\Rightarrow b^4 = e \Rightarrow a^2 = e$ false

Similarly, if $b^2 = a^2$ then

Mult on rt by a^2: $b^2a^2 = a^4 = a \overset{\text{mult by } a^2 \text{ on rt}}{\Rightarrow} b^2 a = e$

$\Rightarrow a^2 = e$

$b^2 = e \Leftarrow$

$\therefore\ G = \{ b^j a^k : b^2 = e,\ a^3 = e,\ ab = ba^2 \}$

$\cong S_3$ via. $b \mapsto (1,2)$
$a \mapsto (1,2,3)$

$ab \mapsto (1,2,3)(1,2) = (1,3)$
$ba^2 \mapsto (1,2)(1,3,2) = (1,3)$ ✓

$\therefore$ If G has order 6, $G \cong S_3$ or
$G \cong \mathbb{Z}_6$

$\boxed{\text{Example}}$ Let H be the subgroup of $S_7 = G$ generated by $(1,2,3,4)$ and $(5,6)$.

① Find $C_G(H)$

② Find $N_G(H)$

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① Since $(1,2,3,4)$ & $(5,6)$ commute,

$H = \{ e, \underset{\tau}{(1,2,3,4)}, (1,3)(2,4), (1,4,3,2), \overset{\alpha}{(5,6)}, (1,2,3,4)(5,6), (1,3)(2,4)(5,6), (1,4,3,2)(5,6) \}$

$= \{ \tau^j \alpha^k : \tau^4 = e = \alpha^2, \alpha\tau = \tau\alpha \} = \{ e, \tau, \tau^2, \tau^3, \alpha, \tau\alpha, \tau^2\alpha, \tau^3\alpha \} = \langle \alpha, \tau : \tau^4 = e, \alpha^2 = e, \alpha\tau = \tau\alpha \rangle$ abelian $\alpha\tau = \tau\alpha$, $\alpha^2 = e$, $\tau^4 = e$

$H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

If $\sigma \in C_G(H) \Leftrightarrow \sigma h \sigma^{-1} = h \quad \forall h \in H.$

$$\Leftrightarrow \begin{array}{l} \sigma \tau \sigma^{-1} = \tau \\ \sigma \alpha \sigma^{-1} = \alpha , \end{array}$$

$$\Leftrightarrow \begin{array}{l} \sigma (1,2,3,4) \sigma^{-1} = (1,2,3,4) \\ \sigma (5,6) \sigma^{-1} = (5,6) \end{array}$$

$$\Leftrightarrow \begin{array}{l} (\sigma(1), \sigma(2), \sigma(3), \sigma(4)) = (1,2,3,4) \\ (\sigma(5), \sigma(6)) = (5,6) \end{array}$$

$\sigma \tau \sigma^{-1} = \tau$

$\Rightarrow \sigma \tau^2 \sigma^{-1} = \sigma \tau \sigma^{-1} \sigma \tau \sigma^{-1}$

$= \tau^2$

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to be continued...